

BEYOND THE ‘PENTAGON IDENTITY’.

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Abstract. An algebraical background of the Lattice Conformal Field Theory is refined with the help of a novel q -exponential identity.

It is commonly believed [GR] that the function

$$s(x) = \prod_{n \geq 0} (1 - xq^{2n+1})$$

is a q -world counterpart of the exponential function. It means that as soon as u and v make a Weyl pair

$$uv = q^2vu$$

the q -exponents of them behave just like ordinary exponents of commuting arguments do:

$$s(u)s(v) = s(u+v).$$

Recently [FV] added a missing ‘reversed’ multiplication rule

$$s(v)s(u) = s(u+v-qvu)$$

to the collection of its properties. This time let me present another identity

$$s(v)s(u^{-1})s(u)s(v) = s(u^{-1})s(v)s(u)$$

which is a consequence of the two multiplication rules but apparently has virtues of its own.

So, let me first derive that 7-term identity. Applying the second multiplication rule once and then the first one twice

$$s(v)s(u) = s(u+v-qvu) = s(u+(v-qvu)) = s(u)s(v-qvu) = s(u)s(-qvus(v))$$

we soon come to the 5-term identity[†]

$$s(v)s(u) = s(u)s(-qvus(v))$$

which in turn brings us, again in three steps[‡], to the 7-term one:

$$\begin{aligned} \underline{s(v)s(u^{-1})} \underline{s(u)s(v)} &= s(u) \underline{s(-qvus(v)s(u^{-1})s(v))} \\ &= \underline{s(u)s(u^{-1})} \underline{s(-qvus(v))} = s(u^{-1})s(v)s(u). \end{aligned}$$

One obvious advantage of the 7-term identity, comparing to the 5-term one and the multiplication rules themselves, is that we can now produce a closed set of commutation relations (four nontrivial ones, six in total)

$$\begin{array}{ll} s_2^+ s_1^- s_1^+ s_2^+ = s_1^- s_2^+ s_1^+ & s_2^- s_1^+ s_1^- s_2^- = s_1^+ s_2^- s_1^- \\ s_1^+ s_2^+ s_2^- s_1^+ = s_2^+ s_1^+ s_2^- & s_1^- s_2^- s_2^+ s_1^- = s_2^- s_1^- s_2^+ \\ s_1^+ s_1^- = s_1^- s_1^+ & s_2^+ s_2^- = s_2^- s_2^+ \end{array}$$

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[†]this ‘pentagon’ thing leads already its own life, under the banner ‘Quantum dilogarithm identity’[FK]

[‡]prior to every step I underline the part which is going to be treated

involving just four q -exponents

$$s_1^\pm = s(u^{\pm 1}) \quad s_2^\pm = s(v^{\pm 1}).$$

According to the lattice way of thinking one Weyl pair is good for a lattice of just two sites. For a longer lattice one employs a sort of lattice ‘free field’: an algebra where every ‘nearest neighbours’ pair w_n, w_{n+1} of its N generators w_1, w_2, \dots, w_N is like a Weyl pair

$$w_n w_{n+1} = q^2 w_{n+1} w_n \quad 1 \leq n \leq N-1$$

while all other pairs just commute

$$w_m w_n = w_n w_m \quad |m - n| > 1.$$

For $2N$ q -exponents available

$$s_n^\pm = s(w_n^{\pm 1})$$

$4(N-1)$ nontrivial commutation relations emerge

$$s_{n+1}^\pm s_n^\mp s_n^\pm s_{n+1}^\pm = s_n^\mp s_{n+1}^\pm s_n^\pm s_n^\mp \quad s_n^\mp s_{n+1}^\mp s_{n+1}^\pm s_n^\mp = s_{n+1}^\mp s_n^\mp s_n^\pm s_n^\pm.$$

They are complemented by a bunch of trivial ones

$$s_m s_n = s_n s_m \quad |m - n| \neq 1$$

where s_n means either s_n^+ or s_n^- . Meet a brand new discrete group. Indeed, we can now dispose of the free field and regard s ’s as just generators obeying only the above set of commutation relations.

Of course, the crucial question is whether or not the 7-term identity is all we really want to know about the q -exponent. Apparently it is, at least as far as the Lattice CFT [FV] is concerned. First come the braids. The elements

$$b_n = s_n^+ s_n^- \\ b_m b_n = b_n b_m \quad |m - n| > 1$$

prove to obey the Artin’s commutation relations:

$$\begin{aligned} b_n b_{n+1} b_n &= s_n^- \underline{s_n^+ s_{n+1}^+ s_{n+1}^- s_n^+ s_n^-} = \underline{s_n^- s_{n+1}^+ s_n^+ s_{n+1}^- s_n^-} \\ &= s_{n+1}^+ s_n^+ \underline{s_n^- s_{n+1}^- s_{n+1}^+ s_n^-} \\ &= s_{n+1}^+ \underline{s_n^+ s_{n+1}^- s_n^- s_{n+1}^+} = s_{n+1}^+ s_{n+1}^- s_n^+ s_n^- s_{n+1}^+ s_{n+1}^- = b_{n+1} b_n b_{n+1}. \end{aligned}$$

This is indeed the braid group B_{N+1} . It is however remains to see what the ‘twisted’ set-up

$$\varsigma_n = s_n^- s_{n+1}^+$$

can do. Fortunately, it delivers:

$$\begin{aligned} \varsigma_{n+1} \varsigma_{n-1} \varsigma_n \varsigma_{n+1} &= s_{n+1}^- \underline{s_{n+2}^+ s_{n-1}^- s_n^+ s_n^- s_{n+1}^+ s_{n+1}^- s_{n+2}^+} \\ &= \underline{s_{n+1}^- s_{n-1}^- s_n^+ s_n^- s_{n+1}^-} s_{n+2}^+ s_{n+1}^+ \\ &= s_{n-1}^- s_n^+ s_{n+1}^- s_n^- s_{n+2}^+ s_{n+1}^+ = \varsigma_{n-1} \varsigma_{n+1} \varsigma_n. \end{aligned}$$

Similarly,

$$\begin{aligned} \varsigma_{n-1} \varsigma_n \varsigma_{n+1} \varsigma_{n-1} &= \underline{s_{n-1}^- s_n^+ s_{n+1}^- s_{n+1}^+ s_{n+1}^- s_{n+2}^+ s_{n-1}^-} s_n^+ \\ &= s_n^- s_{n-1}^- \underline{s_n^+ s_{n+1}^+ s_{n+1}^- s_{n+2}^+ s_n^+} \\ &= s_n^- s_{n-1}^- s_{n+1}^+ s_n^+ s_{n+1}^- s_{n+2}^+ = \varsigma_n \varsigma_{n-1} \varsigma_{n+1}. \end{aligned}$$

So, we end up with yet another group and there is a good reason to call (a cyclic version of) this one a ‘lattice Virasoro algebra’ or maybe a ‘discrete conformal group’. This issue, as well as that of YangBaxterization, will be discussed in detail elsewhere.

Anyway, the group

$$\varsigma_{n+1}\varsigma_{n-1}\varsigma_n\varsigma_{n+1} = \varsigma_{n-1}\varsigma_{n+1}\varsigma_n$$

$$\varsigma_{n-1}\varsigma_n\varsigma_{n+1}\varsigma_{n-1} = \varsigma_n\varsigma_{n-1}\varsigma_{n+1}$$

$$\varsigma_m\varsigma_n = \varsigma_n\varsigma_m \quad |m - n| > 2$$

seems to be the most valuable outcome of those q -manipulations. It looks like a close relative to the braid group

$$b_nb_{n+1}b_n = b_{n+1}b_nb_{n+1}$$

$$b_mb_n = b_nb_m \quad |m - n| > 1$$

for despite of their obvious differences they still share some key features. One striking similarity between them is how a single generator goes through long enough ‘ordered’ words:

$$(b_mb_{m+1} \dots b_n)b_k = b_m \dots (b_kb_{k+1}b_k) \dots b_n$$

$$= b_m \dots (b_{k+1}b_kb_{k+1}) \dots b_n = b_{k+1}(b_mb_{m+1} \dots b_n)$$

$$(\varsigma_m\varsigma_{m+1} \dots \varsigma_n)\varsigma_k = \varsigma_m \dots \varsigma_{k-1}(\varsigma_k\varsigma_{k+1}\varsigma_{k+2}\varsigma_k) \dots \varsigma_n$$

$$= \varsigma_m \dots \varsigma_{k-1}(\varsigma_{k+1}\varsigma_k\varsigma_{k+2}) \dots \varsigma_n = \varsigma_m \dots (\varsigma_{k-1}\varsigma_{k+1}\varsigma_k)\varsigma_{k+2} \dots \varsigma_n$$

$$= \varsigma_m \dots (\varsigma_{k+1}\varsigma_{k-1}\varsigma_k\varsigma_{k+1})\varsigma_{k+2} \dots \varsigma_n = \varsigma_{k+1}(\varsigma_m\varsigma_{m+1} \dots \varsigma_n).$$

Of course, the similarity can not remain this literal for reversely ordered words but it appears no less amusing. While in the braid group this is again a one-step translation

$$(b_nb_{n-1} \dots b_m)b_{k+1} = b_k(b_nb_{n-1} \dots b_m),$$

in ς ’s it is a translation by two steps at once:

$$(\varsigma_n\varsigma_{n-1} \dots \varsigma_m)\varsigma_{k+1} = \varsigma_n \dots \varsigma_{k+1}(\varsigma_k\varsigma_{k-1}\varsigma_{k+1}) \dots \varsigma_m$$

$$= \varsigma_n \dots \varsigma_{k+1}(\varsigma_{k-1}\varsigma_k\varsigma_{k+1}\varsigma_{k-1}) \dots \varsigma_m = \varsigma_n \dots (\varsigma_{k+1}\varsigma_{k-1}\varsigma_k\varsigma_{k+1})\varsigma_{k-1} \dots \varsigma_m$$

$$= \varsigma_n \dots (\varsigma_{k-1}\varsigma_{k+1}\varsigma_k)\varsigma_{k-1} \dots \varsigma_m = \varsigma_{k-1}(\varsigma_n\varsigma_{n-1} \dots \varsigma_m).$$

Not really proving anything, these simple tests at least give a hope that the new group is only about as ‘large’ as the braid group. And if this is indeed true, it might find a spectrum of applications reaching far beyond our modest Lattice CFT.

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